

On the Energy Inequality on a Class of Poisson Manifolds

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Abstract

This paper establishes a Poisson analogy of energy capacity inequality on the standard Poisson space. With the help of the leaf structure of the Poisson space and the accurate estimate of the Poisson capacity and the Poisson type Hofer norm, the inequality between the Poisson capacity and the Poisson energy is proved.

Keywords

Poisson energy, Poisson capacity, Hamiltonian diffeomorphism.

1. Introduction

This paper is devoted to establishing an energy capacity inequality on a class of Poisson manifolds. In symplectic geometry, the energy capacity inequality plays an important role in studying bi-invariant metrics on the groups of Hamiltonian diffeomorphisms. Hofer firstly defined a surprising invariant metric on the group of compactly supported Hamiltonian diffeomorphisms of $(\mathbb{R}^{2n}, \omega_0)$ (ω_0 is the standard symplectic form)[6], Viterbo also gave an invariant metric by the method of generating functions[10]. Polterovich, Lalonde and McDuff generalized this metric on symplectic manifold[12,13]. But the non-degeneracy of this metric is not easy to prove. Symplectic capacity is another very important invariant, the relationship between symplectic capacity and the Hofer norm is very crucial in the studying of the Hofer geometry. There are many famous capacities on symplectic manifold, I. Ekeland, H. Hofer defined a class of symplectic capacities for subsets of standard symplectic space [2,3], Gromov defined a capacity named the symplectic width[9], H. Hofer and E. Zehnder defined symplectic capacities on general symplectic cases [4], C. Liu and Q. Wang considered the group action on the symplectic manifolds and defined the symmetrical symplectic capacity [8], they also studied the applications of symplectic capacities on symmetrical Hamiltonian system.

Poisson manifold is a generalization of symplectic manifold. There are many mathematicians such as Poisson, Marsden, Weinstein and so on, who have made great contributions in Poisson geometry [1]. We call a manifold M Poisson, if there exists a Poisson bracket $\{ \}$ on the smooth functions of the manifold, and for any smooth function f, g, h we have the following identities:

$$\begin{aligned}\{f, g\} &= -\{g, f\} \\ \{f, gh\} &= g\{f, h\} + h\{f, g\} \\ \{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} &= 0\end{aligned}\tag{1}$$

In symplectic geometry, there are many important concepts such as symplectic diffeomorphism, symplectic Hamiltonian flow, symplectic energy, symplectic vector and so on. Naturally we have similar notations in the studying of Poisson geometry, such as Poisson type diffeomorphism, Poisson type vector, Poisson type Hamiltonian flow. Due to the lackness of the variational structure in the Poisson case, the research of Hofer norm and the capacity theory is not very

easy. Recently, Sun and Zhang introduced the Hofer type norm in the Poisson geometry with the assumption that the restriction of the Hamiltonian function on the symplectic leaves are well defined[14], T. Rybicki considered sufficient conditions to guarantee the existence of Hofer type norm[15]. Sun and Liu established the notation of Poisson capacity on a class of Poisson manifold[16]. The relation between the Poisson energy and Poisson capacity is an interesting question, for all we know, there are few results about this. In this paper we will give a Poisson analogy of the energy capacity inequality on a class of Poisson manifolds. The main results of this paper are the following:

Theorem 1. Suppose ψ is the Hamiltonian diffeomorphism on the standard Poisson space $(\mathbb{R}^n, \{\cdot, \cdot\}_0)$, the energy capacity inequality holds:

$$\sup \{c_0(U) \mid \psi(U) \cap U = \emptyset\} \leq E(\psi) \quad (2)$$

2. Preliminaries

In this section, we will introduce some elementary notations and properties in symplectic and Poisson geometry, concrete details and proof can be found in references [1,5,6,7,11]. First we introduce the definitions of Poisson diffeomorphism.

Definition 2. We call a diffeomorphism ϕ of M Poisson, if for any smooth functions f, g on the Poisson manifold $(M, \{\cdot, \cdot\})$, the following equality holds

$$\phi^* \{f, g\} = \{f \circ \phi, g \circ \phi\} \quad (3)$$

If a diffeomorphism ϕ on a symplectic manifold keeps the symplectic form, we call the diffeomorphism symplectic. Next we introduce the definition of symplectic capacity.

Definition 3[5]. A symplectic capacity is a function on symplectic manifold (M, ω) , and satisfies the following three properties:

$$c(M, \omega) \leq c(N, \tau) \quad (4)$$

If there exists a symplectic embedding $\varphi: (M, \omega) \rightarrow (N, \tau)$. For all $\alpha \neq 0$,

$$c(M, \alpha\omega) = |\alpha| c(M, \omega) \quad (5)$$

For the open unit ball $B(1)$ and the symplectic cylinder $Z(1)$ in the standard space $(\mathbb{R}^{2n}, \omega_0)$,

$$c(B(1), \omega_0) = c(Z(1), \omega_0) = \pi \quad (6)$$

The ball and the cylinder in $(\mathbb{R}^{2n}, \omega_0)$ is defined as following:

$$\begin{aligned} B(r) &= \{(x, y) \mid |x|^2 + |y|^2 < r^2\} \\ Z(r) &= \{(x, y) \mid |x_1|^2 + |y_1|^2 < r^2\} \end{aligned} \quad (7)$$

Here $(x, y) \in \mathbb{R}^{2n}$ are the symplectic coordinates.

Next, we introduce the definition of Hofer norm and the symplectic energy. For a Hamiltonian function $H(t, x) \in C^\infty([0, 1] \times M, \mathbb{R})$, we define the oscillation as

$$\text{osc}(H_t) = \max_x H(t, x) - \min_x H(t, x) \quad (8)$$

The pseudo-norm is defined as

$$\|H_t\| = \int_0^1 \text{osc}(H_t) dt \quad (9)$$

Definition 4[5]. For a Hamiltonian diffeomorphism φ , we define its energy as

$$E(\varphi) = \inf\{\|H_t\| \mid \phi_H^1 = \varphi\} \quad (10)$$

We remark that it is difficult to extend this norm to the Poisson case directly, since the proof in the symplectic case is variational in nature, but we can not find the variational structure in the Poisson case easily. We need more assumptions on the Poisson manifold.

Definition 5[16]. A manifold is type S if it satisfies the following: the Poisson manifold is closed or open, and all the symplectic leaves are closed or open. If the Poisson manifold is open, we assume that all the Hamiltonian functions are compactly supported and all the symplectic leaves are open or closed, and the Hamiltonian functions restricted to the leaves are still compactly supported when the symplectic leaves are open.

Definition 6[11]. Let $(M, \{\})$ be a Poisson manifold, and Π be a Poisson bi-vector, we define the sharp map by:

$$\begin{aligned} \# \Pi : T^*M &\rightarrow TM \\ \#_{\Pi, x}(x, \alpha_x) &= (x, i_{\alpha_x} \Pi(x)), \quad \alpha_x \in T_x^*M \end{aligned} \quad (11)$$

According to the assumptions and the existence theorem of the differential equation, the solution of the following equation exists

$$\begin{aligned} \dot{x} &= X_{H_t}(t, x) \\ X_{H_t} &= \{\bullet, H_t\} \end{aligned} \quad (12)$$

Definition 7[11]. The Hamiltonian function is complete if the flow of the vector generated by the function globally exists. A Poisson manifold is complete if all the Hamiltonian functions are complete.

Definition 8[11]. Let $(M, \{\})$ be a Poisson manifold, and Π be a Poisson bi-vector, we define the sharp map by:

$$\begin{aligned} \# \Pi : T^*M &\rightarrow TM \\ \#_{\Pi, x}(x, \alpha_x) &= (x, i_{\alpha_x} \Pi(x)), \quad \alpha_x \in T_x^*M \end{aligned} \quad (13)$$

With the help of above definitions, we can introduce the Poisson capacity.

Definition 9[16]. The standard Poisson space $(\mathbb{R}^n, \{\})_0$ is defined as following:

If $n = 2m$, the standard Poisson manifold is just the standard symplectic space $(\mathbb{R}^{2m}, \omega_0)$.

(2) If $n = 2m+1$, for every point in the 1dimensional space $\mathbb{R}$, the symplectic leaf is $2m$ dimensional subspace contained the point, the restriction of the Poisson bracket on symplectic space is the usual bracket generated by the standard symplectic form ω_0 , the restriction of the Poisson bracket on the left space $\mathbb{R}$ is 0.

Definition 10[16]. A Poisson capacity is a map on the sets of Poisson manifold, it is a nonnegative number or infinity satisfying the following properties:

$$c(M, \{\}) \leq c(N, \{\}) \quad (14)$$

If there exists a Poisson embedding $\varphi : (M, \{\}) \rightarrow (N, \{\})$. For all real number $\alpha \neq 0$,

$$c(M, \alpha \{\}) = |\alpha| c(M, \{\}) \quad (15)$$

For the open unit ball $B(1)$ and the Poisson cylinder $Z(1)$ in the standard space $\mathbb{R}^n (n \geq 2)$, $c(B(1), \{ \}_{0})$ and $c(Z(1), \{ \}_{0})$ are finite.

3. Proof of Main Results

In this part we can give the proof of the energy capacity inequality on Poisson manifold. We first recall the construction of the Poisson capacity. By the assumptions of the Poisson manifold, the restriction of the Hamiltonian function and Hamiltonian diffeomorphism on the symplectic leaves are well done. The Poisson capacity is defined as following

$$c(M) = \sup_{l_\alpha} c_s(l_\alpha) \quad (16)$$

Here $l_\alpha \in M$ are the symplectic leaves, c_s is the symplectic capacity on the symplectic manifold. Lemma1[16]. On standard Poisson space $(\mathbb{R}^n, \{ \}_{0})$, the function $c((\mathbb{R}^n, \{ \}_{0})) = \sup_{l_\alpha} c_s(l_\alpha)$ is a Poisson capacity.

$$c(M) = \sup_{l_\alpha} c_s(l_\alpha M) \quad (17)$$

Proof of Theorem 1:

In order to prove the energy capacity inequality, we need more carefully estimate about the capacity.

Case 1: if $n = 2m$, the standard Poisson manifold is just the standard symplectic space $(\mathbb{R}^{2m}, \omega_0)$, then the energy capacity inequality holds obviously, this is just the symplectic energy capacity inequality.

Case 2: if $n = 2m+1$, we can let the Poisson coordinates be $(x_1, x_2, \dots, x_m, y_1, y_2, \dots, y_m, z_{2m+1})$, according to the assumptions on the standard Poisson space, the symplectic leaves can be viewed as parallel $2m$ dimensional subspaces, each subspace is a standard $2m$ symplectic space. For every Hamiltonian diffeomorphism ψ , suppose the restriction of the Hamiltonian diffeomorphism ψ on the symplectic leaf is ψ_α , by the symplectic energy capacity inequality, we have the following inequality on each symplectic leaf

$$\sup \{c_0(U) \mid \psi_\alpha(U) \cap U = \emptyset\} \leq E(\psi_\alpha) \quad (18)$$

On the other hand, the definition of the Poisson energy implies that

$$E(\psi_\alpha) \leq E(\psi) \quad (19)$$

Combine the above inequalities, we have

$$c_0(U) = \sup_{l_\alpha} c_s(l_\alpha U) \leq E(\psi_\alpha) \leq E(\psi) \quad (20)$$

So we have the energy capacity inequality

$$\sup \{c_0(U) \mid \psi(U) \cap U = \emptyset\} \leq E(\psi) \quad (21)$$

4. Conclusion

In this paper we establish the relationship between the Poisson capacity and the Poisson energy on the standard Poisson space. Using the geometry structure of the Poisson manifold, we reduce the problem to the symplectic leaves on the standard symplectic space, we prove that the energy capacity inequality holds on the standard Poisson space. This inequality will play an important role in the related topics in Poisson dynamics, we will continue to study other related questions in the future.

Acknowledgements

This research was supported by National Natural Science Foundation of China (11701149) and Foundation of Henan University of Technology(2015QNJH19).

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